ME 201/APh 250, Homework 1:

Assigned: Friday, Apr 5, 2019
Due: Friday, Apr 12, 2019

N&C 2.27:
(a)

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} X \otimes Z &= \begin{bmatrix} 0.Z & 1.Z \\ 1.Z & 0.Z \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \end{aligned}$$

(b)

$$I \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(c)

$$X \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(d)No. $X \otimes I \neq I \otimes X$

N&C 2.41:
We use i=1, j =2 as examples.

$$\begin{aligned} \{\sigma_1, \sigma_2\} &= \sigma_1\sigma_2 + \sigma_2\sigma_1 \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\sigma_1^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similar computations can be used to verify the remaining relations.

N&C 2.60:

$$\begin{aligned} ||\overrightarrow{\nu} \cdot \overrightarrow{\sigma} - \lambda \hat{I}|| &= \det\left\{ \begin{vmatrix} \nu_3 - \lambda & \nu_1 - i\nu_2 \\ \nu_1 + i\nu_2 & -\nu_3 - \lambda \end{vmatrix} \right\} \\ &= (\nu_3 - \lambda)(-\nu_3 - \lambda) - (\nu_1 - i\nu_2)(\nu_1 + i\nu_2) \\ &= \lambda^2 - \nu_1^2 - \nu_2^2 - \nu_3^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lambda &= \pm\sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2} \\ &= \pm 1 \end{aligned}$$

Therefore, the eigenvalues are $\pm$ 1.

Consider $\lambda = 1$ and solve for normalized eigenvector $|+\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ such that $\overrightarrow{\nu} \cdot \overrightarrow{\sigma} \, |+\rangle = +1 \, |+\rangle$. We have:

$$\begin{aligned} \overrightarrow{\nu} \cdot \overrightarrow{\sigma} \, |+\rangle &= \begin{pmatrix} \nu_3 a + b(\nu_1 - i\nu_2) \\ (\nu_1 + i\nu_2)a - \nu_3 b \end{pmatrix} \\ &= \begin{pmatrix} a \\ b \end{pmatrix} \end{aligned}$$

Comparing component wise, we get $a = \frac{\nu_1 - i\nu_2}{1-\nu_3} b$ and using normalization condition $|a|^2 + |b|^2 = 1$, one gets $a = \frac{\nu_1 - i\nu_2}{1-\nu_3}\sqrt{\frac{1-\nu_3}{2}}$ and $b = \sqrt{\frac{1-\nu_3}{2}}$.

$$\begin{aligned} P_+ &= |+\rangle \langle +| \\ &= \frac{1}{2} \begin{bmatrix} \frac{\nu_1^2 + \nu_2^2}{1-\nu_3} & v_1 - i\nu_2 \\ v_1 + i\nu_2 & 1 - \nu_3 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \frac{\nu_1^2 + \nu_2^2 + \nu_3^2 - \nu_3^2}{1-\nu_3} & v_1 - i\nu_2 \\ v_1 + i\nu_2 & 1 - \nu_3 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 + \nu_3 & v_1 - i\nu_2 \\ v_1 + i\nu_2 & 1 - \nu_3 \end{bmatrix} \\ &= \frac{1}{2} \hat{I} + \overrightarrow{\nu} \cdot \overrightarrow{\sigma} \end{aligned}$$

Similar argument holds for $P_- = |-\rangle\langle-|$.

N&C 2.61:

The probability is given by:

$$\begin{aligned}\langle 0|P_+|0\rangle &= \frac{1}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1+\nu_3 & v_1 - i\nu_2 \\ v_1 + i\nu_2 & 1-\nu_3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \\ &= \frac{1}{2}(1+\nu_3)\end{aligned}$$

$|0\rangle$ will collapse into the state $|+\rangle$.

N&C 2.66:

The average value is given by:

$$\begin{aligned}\frac{1}{2}(((\langle 00| + \langle 11|)(X_1Z_2)(|00\rangle + |11\rangle)) &= (\langle 00| X_1Z_2 |00\rangle + \langle 00| X_1Z_2 |11\rangle + \langle 11| X_1Z_2 |00\rangle + \langle 11| X_1Z_2 |11\rangle)/2 \\ &= (\langle 0| X_1 |0\rangle \langle 0| Z_2 |0\rangle + \langle 0| X_1 |1\rangle \langle 0| Z_2 |1\rangle) \\ &\quad + \langle 1| X_1 |0\rangle \langle 1| Z_2 |0\rangle + \langle 1| X_1 |1\rangle \langle 1| Z_2 |1\rangle)/2 \\ &= (\langle 0|1\rangle \langle 0|0\rangle + \langle 0|0\rangle (-\langle 0|1\rangle) + \langle 1|1\rangle \langle 1|0\rangle + \langle 1|0\rangle (-\langle 1|0\rangle))/2 \\ &= 0\end{aligned}$$

N&C 2.71:

From spectral decomposition theorem,

$$\rho = \sum_j \lambda_j |j\rangle\langle j|$$

where λ_j are real and non-negative and $\sum_j \lambda_j = 1$. From this, we can deduce $0 \le \lambda_j \le 1$. Now, $\rho^2 = \sum_j \lambda_j^2 |j\rangle\langle j|$ and $\text{Tr}\{\rho^2\} = \sum_j \lambda_j^2 \le \sum_j \lambda_j = 1$. Since $0 \le \lambda_j \le 1$, $0 \le \lambda_j^2 \le 1$. For $\sum_j \lambda_j^2$ to be 1, all but one of λ_j have to vanish and the remaining one is 0. This implies the state is $p = |j\rangle\langle j|$; it is a pure state. If the state is pure, then $\rho = |j\rangle\langle j|$ and $\rho^2 = |j\rangle\langle j|$. Its trace is clearly one.

N&C 2.79:

(a) It is clear that $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ is a valid Schmidt decomposition.

(b) Let $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Then,$|\psi\rangle = 1\,|{+}{+}\rangle + 0\,|{-}{-}\rangle$.

(c) Let $|\psi\rangle = \sum_{ij} A_{ij} |i\rangle |j\rangle$We construct the matrix $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix}$. We note that if $A = USV$, then $|\psi\rangle = \sum_{i=0}^{d-1} s_i (\sum_j^{d_A-1} u_{j,i} |j\rangle_A) \otimes (\sum_k^{d_B-1} u_{i,k} |k\rangle_B)$. Performing SVD(use python/mathematica/matlab etc.), we get:

$$A = \begin{bmatrix} -0.851 & -0.526 \\ -0.526 & 0.851 \end{bmatrix} \begin{bmatrix} 0.934 & 0.357 \\ \frac{1}{\sqrt{3}} & 0 \end{bmatrix} \begin{bmatrix} -0.851 & -0.526 \\ 0.526 & -0.851 \end{bmatrix}$$

Then,

$$|0_A\rangle = \begin{pmatrix} -0.85065081 \\ -0.52573111 \end{pmatrix}$$
$$|1_A\rangle = \begin{pmatrix} -0.52573111 \\ 0.85065081 \end{pmatrix}$$
$$|0_B\rangle = \begin{pmatrix} -0.85065081 \\ -0.52573111 \end{pmatrix}$$
$$|1_B\rangle = \begin{pmatrix} 0.52573111 \\ -0.85065081 \end{pmatrix}$$