Microwave Noise in Semiconductor Devices

Module 2

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1 Stochastic signals in time and frequency domain

[Ambrozy Chap 3, all figures from here]

We have defined a stochastic process as a single parameter assembly of random variables over a set of times. The RV, 　　　, could represent voltage or current and be regarded as a discrete or continuous function of time.

In many cases, the function describing e.g. voltage is given by a formula with a precise time dependence - a deterministic function. Often a few parameters - amplitude, frequency, etc - fully specify the past and future of the signal over some time interval.

A stochastic signal does not allow knowledge of the next instantaneous value! Only probabilities. In practice, some correlation might exist owing to small but finite time constants (e.g. transit time of an electron across a gap). For very small time intervals, the instantaneous values are not independent of each other. This property is known as autocorrelation.

First, we need to describe deterministic and stochastic signals in the time and frequency domain.

2   Deterministic and stochastic signals

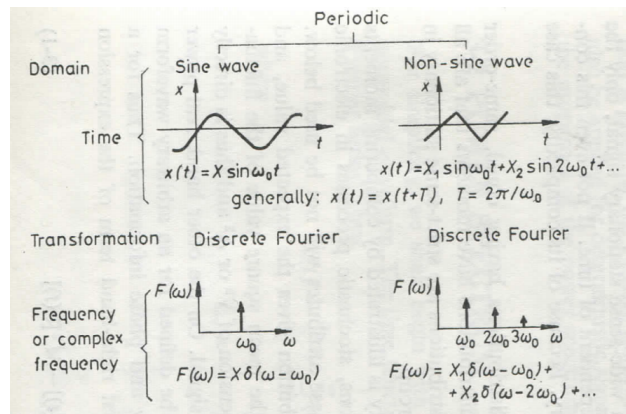**Simplest deterministic signal** - cosines and sines with integral multiples of frequency

Figure 1: Schematic of various types of signals.

To get to frequency domain → discrete Fourier transform.

Almost periodic signal - superposition of sines and cosines

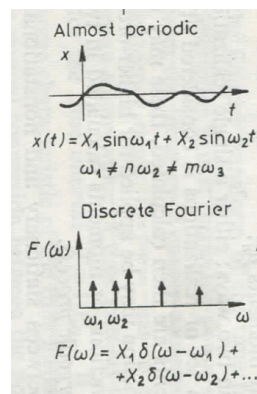with various frequencies satisfying

Figure 2: Almost periodic signals.

Such a function repeats only after infinite time. They also have a discrete frequency domain

spectrum.

Transients - functions for which   if   . We use a Fourier transform

to get a continuous frequency spectrum. Although the Fourier transform might not converge,

we use it with caution anyway.

Single

x

t

$x = x(t)$

+...

Fourier integral

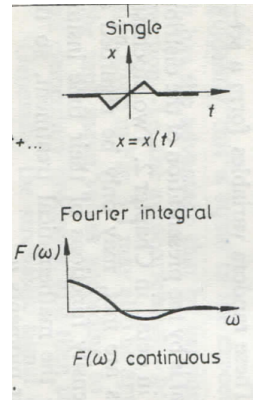$F(\omega)$

ω

$F(\omega)$ continuous

Figure 3: Continuous signal and its Fourier transform.

Stochastic signals are similar to almost periodic signals, but the frequency spectrum of stochastic signals is continuous.

Since we can't predict the amplitude and phase of the signal at an arbitrary time instant, we also don't know the amplitude and phase of the Fourier transform. We need another description.

Let's figure it out. We can define a stochastic process by a function ____ , just as for a deterministic process. ____ may have real or complex values (or vectors). If we get a specific function ____ during a trial, that is called a realization of the stochastic process.

On the other hand, say we fix time ____ . ____ may have different values during consecutive trials so that ____ at fixed ____ is also a random variable. This set of random variables is called an ensemble.

We can characterize a stochastic signal by its distribution function (as in Module 1). We can say two stochastic functions are equivalent if the ensemble density functions are equal. Often in place of density functions we use their moments ____ .

So we have two ways to define a density function for a stochastic process:

a single realization as ____ varies (first moment = ____ average);

set of realizations at a fixed time instant (first moment = ____ average);

We can do the same for higher-order moments.

Ergodic process - the case in which the density functions (or moments) calculated these two different ways are the same.

With this way to characterize stochastic processes, we can define

stationary and non-stationary stochastic signals.

| Stationary | Non-stationary |

All moments independent of time.

Ergodic group: contains distributions having equal, same-order moments. Most often, distributions in electronic devices are stationary and ergodic.

Now let's consider the physical meaning of the moments.

First-order - expected value

Central second-order - mean square value of fluctuation $\rightarrow$ variance.

Dimension of second-order moment =   or $\rightarrow$ connected   of the stochastic signal.

Real power is a physical quantity that can be converted to heat for any waveform (frequency and phase information is lost). So the power spectrum - or power contained in various frequency bands - is a physically meaningful and measurable quantity.

Later we will see the link between the time-domain equations and power spectrum is via Fourier transform and autocorrelation function. The ACF is:

This function is the expected value of   for   . It can also be regarded as the DC component of the power in the signal. For   , it also shows the correlation between consecutive time instants.

Intuitively, we should have that the total power in the range must equal the power obtained from the mean square value of time instants for . We will derive Parseval's theorem that states this fact soon.

2.1   Power spectrum

Consider a deterministic (periodic non-sinusoidal) waveform. We can always expand it in a Fourier series:

Using the Euler relation, we get:

Introducing complex coefficients:

So we get   consisting of a constant and vectors rotating at   .

Take   . Note that

For negative frequencies, note that   switches sign, but   does not. So for all , we can use the universal coefficient:

rather than separate coefficients   and   . So the original signal can be expanded as:

With this expansion, we can obtain the mean square value of   . We multiply the Fourier expansion of   by its complex conjugate and average in time:

The period   . For   , the integral term gives zero.

So we get:

Also remember that   and   for the same   are conjugates and hence have equal magnitudes. Therefore,   is even.

Here is an example plot of the Fourier coefficients and double-sided power spectrum versus frequency:

Figure 3-2 Periodic nonsinusoidal signal. (a) Complex Fourier coefficients as a function of discrete $n\omega_0$ frequencies. (b) Double-sided power spectrum.
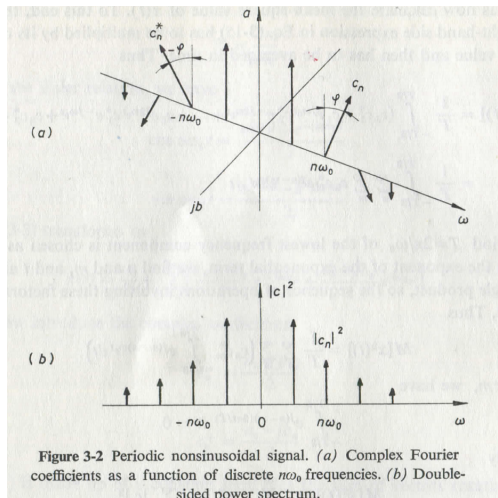
Figure 4:

Note that the dimension of   is   or   , and so the dimension of

is   or . If the given voltage (or current) is dropped across (or flows through) a resistance

, we see we get units of power $\rightarrow$ we have a power spectrum:

Often we just refer to   as the power spectrum.

So we get the theorem: the power of a periodic nonsinusoidal signal can be computed from the Fourier coefficients.

Regarding the negative frequencies: since   is even and we have assumed no DC offset

(   ), we have:

Notation: lower case = two sided power spectrum, upper case = one sided power spectrum (only positive frequencies).

Now consider a deterministic signal with a continuous Fourier spectrum. The discrete spectrum → continuous spectrum   , using:

The function   must satisfy certain requirements.

   1.   should be continuous and differentiable.

   2. Integral of absolute value of   should be finite:

Since (2) is quite restrictive, we can instead require that   is defined on   :

So that the Fourier transform is:
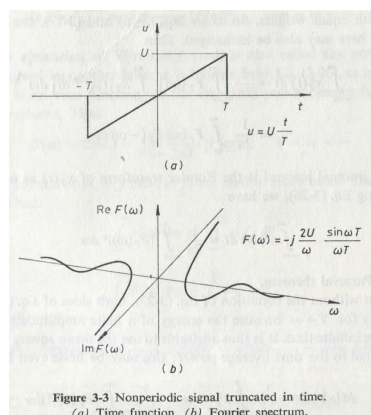
Example: ramp function

Figure 3-3 Nonperiodic signal truncated in time.
(a) Time function. (b) Fourier spectrum.

Figure 5: Ramp function.

Some other nice properties.

Since   is real (e.g. it consists of voltage values from a scope), we can write:

Now rewrite with   rather than   . First term is the same, second term changes sign. So

If we know   , we can get   by inverse Fourier transform:

Now let's do something more interesting by computing the time integral of   . This is proportional to the energy in the signal:

We can switch the order of integrations:

And using the relation at top of page, we get Parseval's theorem:

A subtlety: for signals that have a finite value outside of a time window (e.g. NOT the ramp

function shown before), the power can   as   . We can instead compute the mean

square value = time average power:

Considering functions without the time window restriction (just requiring that   goes

to a finite limiting value for   ):

Taking   to have units of current or voltage, average power is:

      current   voltage

From here we can define a power density:

Note: this definition is for a two-sided power spectrum:

Often another quantity is also referred to as the power density:

Units:

Since   is an even function, we can introduce the one-sided power density:

Finally, we often express the distribution functions in terms of   rather than   .
Using   , we get:

Here is an example of the Fourier spectrum of a signal and its power spectra:
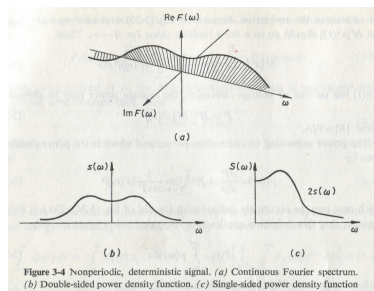

Figure 6: Example Fourier spectrum.

Note that   is a complex function → contains   real numbers for each . However,
has only one real number.   cannot be determined from

What happens for stochastic processes? We can still define a mean square value and hence
obtain the average power. The following result still holds:

The problem that arises for stochastic signals is that   may not exist. Consider a
stationary signal of infinite time duration. Condition (2) is not met since:

Interestingly,   can still be defined. The stationarity of the stochastic process causes

to be a finite, time-independent constant. That means that

is also finite and so the function   is well-behaved in some sense. Note that Dirac deltas could exist in   without affecting this argument.

The upshot is that the power density can be defined for stochastic signals, but sequence of steps used to obtain the power density of a signal of a deterministic signal may not work for a stochastic signal.

Hence we must introduce another description - the autocorrelation function.

2.2   Correlation, autocorrelation

Up till now we have assumed independence of random variables. What if that is not true, and how to measure it?

One way is via a correlation function. Say we have two RVs,   . CF is:

Rewriting:

For   , we have a familiar result:

We now define

[If   have DC components, we can incorporate them easily.]

Now consider the possibility of allowing time shifts between the two functions:

This is the cross-correlation function.

We can apply this definition for a single stochastic process to get

or the autocorrelation function. [We require   ]

If there is no correlation between   for any   ,
then   .

If there is correlation,   will tend to   for   .

For stationary stochastic processes   is independent of   . We are free to
choose the integration interval   .

The above equation is valid for stationary ergodic processes.
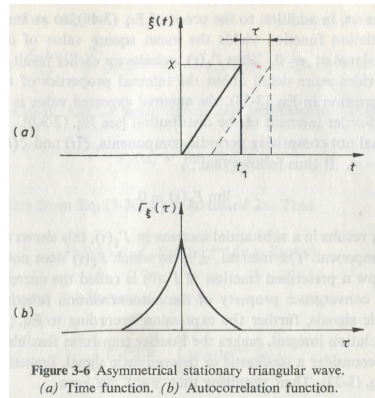
Properties:

 1. (1)

 2. (2)

Figure 3-6 Asymmetrical stationary triangular wave.
(a) Time function. *(b)* Autocorrelation function.

Figure 7: Signal and auto-correlation function.

Proof of (1): For your HW.

Proof of (2):

Related result:　　　　　　　　yields the mean square value of a signal at　　　　　.　　　　thus provides additional information on the stochastic signal.

If a signal is not periodic,　　　　　　　　　　　are independent as　　　　　　　　.
Thus

If　　　increases and an increase in　　　　　　　　is observed, a periodic component is present in the signal.

The interval　　　　　in which　　　　　　　　does not decrease permanently below a given

fraction of $\qquad$ is called the correlation interval.

The convergent property of $\qquad$ has great implications for the Fourier transform. Consider a signal limited to a time window as before. We have the ACF:

Take a Fourier transform of $\qquad$:

Add a factor of $1 = $ $\qquad$:

We can do either integration first. Switch the order:

Define $\qquad$ so that

Note that $\qquad$ depends only on $\qquad$. $\qquad$ is now

Now let's remove the requirement of finite   . Define

Recalling the earlier definition of power spectrum, we see that

Then, if we inverse Fourier transform, we find:

In other words, the ACF   and power spectrum   are related by a

Fourier transform. We have derived the Wiener-Khintchine theorem.

Common forms and notation:

Since   is a real, even function,

Relation to autocorrelation function:

At   :

as we have derived before.

Gaussian stochastic processes are those that have an ensemble density function following a multi-dimensional Gaussian at any fixed time instant. A nice property is that these distributions remain Gaussian even after a linear transformation.

More concretely: take two random variables,   . The values are and . Let   , variance   , and let them follow a two-variable normal distribution   . Then

$$\sigma_x^2 \;=\; M[\xi^2(t)] = M[\xi^2(t+\tau)] = \int_{-\infty}^{\infty} x^2 f(x)dx \tag{1}$$

$$\Gamma_\xi(\tau) \;=\; M[\xi(t)\xi(t+\tau)] = \int\int_{-\infty}^{\infty} x_1 x_2 f(x_1, x_2)dx_1 dx_2 \tag{2}$$

The process is Gaussian if:

$$f(x_1, x_2) \;=\; \frac{1}{2\pi\sigma_x^2\sqrt{1-\rho}} e^{-(x_1^2 - 2\rho x_1 x_2 + x_2^2)/(2\sigma_x^2(1-\rho^2))} \tag{3}$$

$$\rho \;=\; \rho_x(\tau) = \frac{\Gamma_\xi(\tau)}{\sigma_x^2} \tag{4}$$

The common density function of instantaneous values always has a normal distribution, no matter the value of   .   depends on   .

3   Typical autocorrelation functions and their power spectra

3.1   Constant signal

Function:

Autocorrelation function:

This value is also the mean square value.

Power spectrum:

Reminder of Dirac delta properties:

3.2   Sinusoidal signal

Signal:

Autocorrelation function:

Since   is independent of phase   , this result is valid for   as well.

Power spectrum:

It consists of two weighted Dirac deltas symmetric about the origin.

Single-sided power spectrum:

As expected, all spectral power is in one spectral line for this sinusoid.

If we know the time function, getting spectral power is straightforward. For stochastic signals, we generally can measure   . We can then get   but not the original time signal since the phase information is lost.

Consider band-limited white noise with a single-sided power spectrum as:

Autocorrelation function:

Let's consider two cases.

    1.   so that
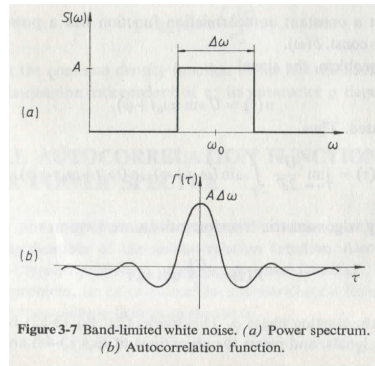
Figure 8:   Band-limited white noise and ACF.

Figure 3-7 Band-limited white noise. *(a)* Power spectrum. *(b)* Autocorrelation function.

So, in the limit of low bandwidths and short correlation times, a stochastic signal can be approximated by a sinusoidal signal of frequency   . Now, previously we found from the Rayleigh distribution that narrow-band noise can be thought of as a carrier with amplitude and phase stochastically modulated.

The above analysis tells us that actually the carrier amplitude can be approximated as constant (in this limit).
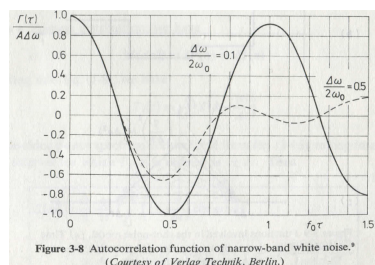

Figure 9: ACF of narrow-band white noise.

Figure 3-8 Autocorrelation function of narrow-band white noise.* *(Courtesy of Verlag Technik, Berlin.)*

General rule: smaller the bandwidth, the more deterministic the signal is ↔ the slower the random phase changes. We will use an approximation based on this later.

2. Say   . Then

As   ,   goes to zero faster. In the limit, we have:

Therefore, the autocorrelation time interval of ideal white noise with infinite bandwidth is zero $\rightarrow$ no correlation between instantaneous values.

3.3   Shot noise

Let's return to our early calculation of shot noise power spectrum.

Here is a sketch of the current fluctuation at time   and   :
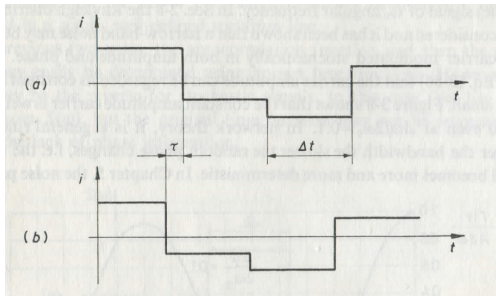

Figure 10: Shot noise current versus time.

In equations:
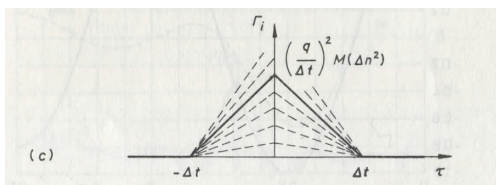
The autocorrelation function is:

Figure 11:   ACF   of shot noise current.

To compute this function,   has to be shifted left or right and the product of instantaneous values has to be averaged for a long time.

Now, the value   , are independent -> the autocorrelation interval is confined to   .

The autocorrelation function of rectangular pulses has a triangle shape.

To get the average height, get   by setting   in ACF.

Average height is then

So that

We already figured out that   has a Poisson distribution. Therefore,

So we get:

We get the single-sided power spectrum in the usual way (remembering that   is defined within only   ).

If   , sinc -> 1, and

Transforming to frequency rather than angular frequency, we get our original result:

The mean square value of the fluctuation in a frequency band   is

This calculation helps us see a limitation on   : it cannot be too small because otherwise

the assumption of independence of current in each time interval is not true.

In the shot noise derivation, we subdivided time into arbitrary intervals and found current pulses due to this subdivision.

Some noise sources naturally have a pulse waveform (e.g. burst noise and processes related to voltage breakdown). A time function of "random telegraph noise" is
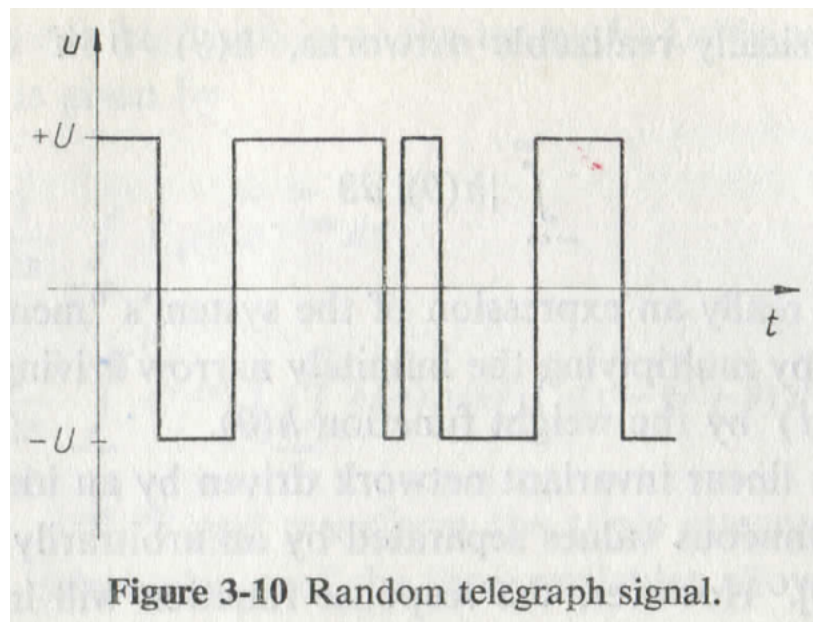

Figure 3-10 Random telegraph signal.

Figure 12: Telegraph noise.

You will see how to obtain the spectral power of this signal in the HW.

4   Response of a linear network driven by a stochastic signal

For small signals, we describe electronic devices as linear networks. In a transistor, several different noise sources can simultaneously exist. In between them can exist resistive and reactive network elements. The ACF and power spectrum that is measured is affected by this network.

The most general relation between the input   and output   of a linear network
with weight function   is

By causality, physical networks have   .
So that the output does not diverge, we require

Now consider an ideal white noise source with independent instantaneous values for an
arbitrarily short time interval (   ). By the linear relation, the
output will contain the driving function value at a time instant, as well as earlier input
values owing to the 'memory' of   . Therefore, the ACF of the driving and response
waveforms are not the same in general.

Let the input driving signal   be stationary and ergodic with ACF   .
We want the ACF   of the response   .

To get it, first compute   :

Recall the definition of ACF:

So we need to integrate in time. We can exchange the order of integrations:

Let's use Fourier transforms to simplify this expression. Recall that

So the Fourier transform of ACF is:

Multiply by   , and exchange order of integrals again:

Since   is real (it is the impulse response of the electrical network consisting of e.g. voltage measurements from a scope), the first two terms are a conjugate pair:

So in the frequency domain, we get the output response simply by multiplication. We often use this equation rather than the time-domain version. Relations:

Example: ACF of the output signal of a single-pole low pass filter. Transfer function:

Input =   , single-sided power spectrum:

Filter output:

ACF:

Schematic figures:

Figure 3-11 White-noise characteristics. *(a)* Power spectrum, without band-limiting. *(b)* Power spectrum, with band-limiting. *(c)* Autocorrelation function, without band-limiting. *(d)* Autocorrelation function, with band-limiting.
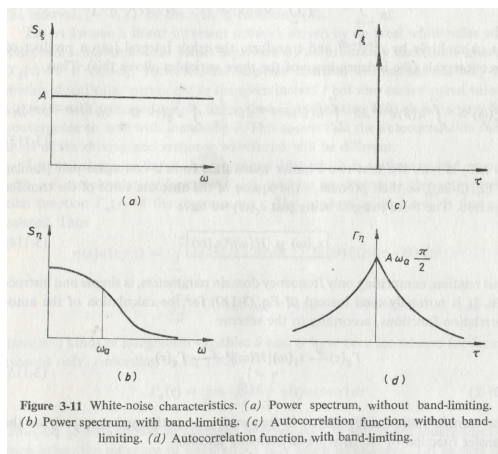
Figure 13: Output through a band-limiting network.

If the input signal has a normal distribution, output signal will also. In fact, a band-limited network output will be closer to normal because output   is a linear combination of instantaneous values weighted by   . So, the sum distribution should approach the normal distribution due to central limit theorem.

In practice, we need the time constant of the network to be larger than the AC interval so that the samples are independent.