ME 201/APh 250, Homework 2:

Assigned: Friday, Apr 12, 2019
Due: Friday, Apr 19, 2019

N&C 4.8:
Let us expand the unitary $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in terms of the set of Pauli matrices $\{I, X, Y, Z\}$(note this set forms a complete basis set of all 2x2 Hermitian matrices). We let $U = a_0 I + a_1 X + a_2 Y + a_3 Z$ where

$$a_0 = \frac{a+d}{2}, a_1 = \frac{b+c}{2}, a_2 = \frac{c-b}{2i}, a_0 = \frac{a-d}{2}$$

Since U is unitary, $U^\dagger U = I$. Then, we have the following list of constraints for a_0, a_1, a_2, a_3

$$\begin{aligned} \|a_0\|^2 + \|a_1\|^2 + \|a_2\|^2 + \|a_3\|^2 = 1 \\ a_0^* a_1 + a_1^* a_0 + i a_2^* a_3 - i a_3^* a_2 = 0 \\ a_0^* a_2 - i a_1^* a_3 + a_2^* a_0 + i a_3^* a_1 = 0 \\ a_0^* a_3 + i a_1^* a_2 - i a_2^* a_1 + a_3^* a_0 = 0 \end{aligned}$$

Let $cos(\frac{\theta}{2}) = \|a_0\|$, from the first constraint, we get $\|a_1\|^2 + \|a_2\|^2 + \|a_3\|^2 = \left\|sin(\frac{\theta}{2})\right\|$ Let

$$\begin{aligned} n_x &= \|a_1\| / \|sin(\theta/2)\| \\ n_y &= \|a_2\| / \|sin(\theta/2)\| \\ n_z &= \|a_3\| / \|sin(\theta/2)\| \end{aligned}$$

Our definition satisfies $n_x^2 + n_y^2 + n_z^2 = 1$. We now define the following:

$$\begin{aligned} a_0 &= e^{i\alpha} cos(\theta/2) \\ a_1 &= -ie^{i\alpha} sin(\theta/2) n_x \\ a_2 &= -ie^{i\alpha} sin(\theta/2) n_y \\ a_3 &= -ie^{i\alpha} sin(\theta/2) n_z \end{aligned}$$

It can be shown these satisfies the remaining constraints. Therefore,

$$\begin{aligned} U &= a_0 I + a_1 X + a_2 Y + a_3 Z \\ &= e^{i\alpha}(cos(\theta/2) I - isin(\theta/2)(n_x X + n_y Y + n_z Z)) \\ &= e^{i\alpha} R_{\vec{n}}(\theta) \end{aligned}$$

N&C 4.17:
We can check that $CNOT = (I \otimes H)(C_Z)(I \otimes H)$

N&C 4.32:

In the discussion in **N&C 2.4.1** from page 99 to 101, throwing away information after measurement results in a density matrix given by:

$$\rho' = \sum_m M_m \rho M_m^\dagger$$

Since we are performing projective measurements, it turns out that $M_m = P_m, M_m^\dagger = P_m^\dagger = P_m = M_m$. Then,

$$\rho' = P_0 \rho P_0 + P_1 \rho P_1$$

To show that $tr_2(\rho) = tr_2(\rho')$, consider $\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \left|\psi_i\right\rangle = \sum_{mn} c_i^{mn} \left|mn\right\rangle$,

$$\begin{aligned}\operatorname{tr}_2 \rho &= \operatorname{tr}_2(\sum_i p_i \sum_{mnrs} c_i^{mn} c_i^{rs\star} \left|mn\right\rangle \left\langle rs\right| \\ &= \sum_i p_i \sum_{mnrs} c_i^{mn} c_i^{rs\star} \left|m\right\rangle \left\langle r\right| \operatorname{tr}(\left|n\right\rangle \left\langle s\right|)\end{aligned}$$

$$\begin{aligned}\operatorname{tr} \rho' &= \operatorname{tr}_2(\sum_i p_i \sum_{mnrs} c_i^{mn} c_i^{rs\star} \left|m\right\rangle \left\langle r\right| (\left\langle 0|n\right\rangle \left\langle s|0\right\rangle \left|0\right\rangle \left\langle 0\right| + \left\langle 0|n\right\rangle \left\langle s|1\right\rangle \left|0\right\rangle \left\langle 1\right| \\ &+ \left\langle 1|n\right\rangle \left\langle s|0\right\rangle \left|1\right\rangle \left\langle 0\right| + \left\langle 1|n\right\rangle \left\langle s|1\right\rangle \left|1\right\rangle \left\langle 1\right|)) \\ &= \operatorname{tr}_2(\sum_i p_i \sum_{mnrs} c_i^{mn} c_i^{rs\star} \left|m\right\rangle \left\langle r\right| [\left\langle 0|n\right\rangle \left\langle s|0\right\rangle + \left\langle 1|n\right\rangle \left\langle s|1\right\rangle] \\ &= \sum_i p_i \sum_{mnrs} c_i^{mn} c_i^{rs\star} \left|m\right\rangle \left\langle r\right| \operatorname{tr}(\left|n\right\rangle \left\langle s\right|)\end{aligned}$$

N&C 4.33:
The circuit is given by

$$\begin{aligned}C &= \frac{1}{\sqrt{2}}(\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \otimes \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}) \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 1 & 0\\1 & 0 & 0 & -1\\0 & 1 & -1 & 0\end{bmatrix}\end{aligned}$$

The action of the circuit on the bell state B_{00} is

$$\begin{aligned}CB_{00} &= C\frac{1}{\sqrt{2}} \begin{bmatrix}1\\0\\0\\1\end{bmatrix} \\ &= \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \\ &= \left|00\right\rangle\end{aligned}$$

Similarly,

$$CB_{01} = |01\rangle\,, CB_{10} = |10\rangle\,, CB_{11} = |11\rangle$$

It is clear our measurement operators are given as:

$$\begin{aligned}
M_1 &= |00\rangle\langle B_{00}| \\
M_2 &= |01\rangle\langle B_{01}| \\
M_3 &= |10\rangle\langle B_{10}| \\
M_4 &= |11\rangle\langle B_{11}|
\end{aligned}$$

Then, the POVM elements are:

$$\begin{aligned}
E_1 &= M_1^\dagger M_1 = |B_{00}\rangle\langle B_{00}| \\
E_2 &= M_2^\dagger M_2 = |B_{01}\rangle\langle B_{01}| \\
E_3 &= M_3^\dagger M_3 = |B_{10}\rangle\langle B_{10}| \\
E_4 &= M_4^\dagger M_4 = |B_{11}\rangle\langle B_{11}|
\end{aligned}$$

They are the projectors form from your Bell states.

Qns 2:

We note that the total system AB is spanned by the orthonormal basis vectors: $\{|00\rangle\,, |01\rangle\,, |10\rangle\,, |11\rangle\}$. In this basis, the wave function can be rewritten as $|\psi\rangle = u_1|00\rangle + u_2|01\rangle + u_3|10\rangle + u_4|11\rangle$. Then, the density matrix is given by:

$$\begin{aligned}
\rho &= |\psi\rangle\langle\psi| \\
&= (u_1|00\rangle + u_2|01\rangle + u_3|10\rangle + u_4|11\rangle)(u_1^\star\langle 00| + u_2^\star\langle 01| + u_3^\star\langle 10| + u_4^\star\langle 11|) \\
&= u_1u_1^\star|00\rangle\langle 00| + u_1u_2^\star|00\rangle\langle 01| + u_1u_3^\star|00\rangle\langle 10| + u_1u_4^\star|00\rangle\langle 11| \\
&+ u_2u_1^\star|01\rangle\langle 00| + u_2u_2^\star|01\rangle\langle 01| + u_2u_3^\star|01\rangle\langle 10| + u_2u_4^\star|01\rangle\langle 11| \\
&+ u_3u_1^\star|10\rangle\langle 00| + u_3u_2^\star|10\rangle\langle 01| + u_3u_3^\star|10\rangle\langle 10| + u_3u_4^\star|10\rangle\langle 11| \\
&+ u_4u_1^\star|11\rangle\langle 00| + u_4u_2^\star|11\rangle\langle 01| + u_4u_3^\star|11\rangle\langle 10| + u_4u_4^\star|11\rangle\langle 11|
\end{aligned}$$

We note that $\mathrm{Tr}_B(|pq\rangle\langle rs|) = \langle q|s\rangle|p\rangle\langle r|$. We note that the partial trace is an inner operator. Therefore, taking partial trace over B, we obtain:

$$\begin{aligned}
\mathrm{Tr}_B(\rho) = u_1u_1^\star|0\rangle\langle 0| &+ u_1u_3^\star|0\rangle\langle 1| \\
+ u_2u_2^\star|0\rangle\langle 0| &+ u_2u_4^\star|0\rangle\langle 1| \\
+ u_3u_1^\star|1\rangle\langle 0| &+ u_3u_3^\star|1\rangle\langle 1| \\
+ u_4u_2^\star|1\rangle\langle 0| &+ u_4u_4^\star|1\rangle\langle 1|
\end{aligned}$$

Qns 3:

Note that $H\left|0\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + \left|1\right\rangle)$ and $H\left|1\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle - \left|1\right\rangle)$. Therefore, $H\left|x\right\rangle = \frac{1}{\sqrt{2}}(\left|0\right\rangle + (-1)^x\left|1\right\rangle)$ where $x \in \{0, 1\}$. We can rewrite this as $H\left|x\right\rangle = \frac{1}{\sqrt{2}}\sum_{z\in\{0,1\}}(-1)^{xz}\left|z\right\rangle$. Then,

$$\begin{aligned}
H^{\otimes n}\left|x_1\right\rangle\left|x_2\right\rangle ... \left|x_n\right\rangle &= (H\left|x_1\right\rangle)(H\left|x_1\right\rangle)...(H\left|x_1\right\rangle) \\
&= \frac{1}{\sqrt{2}}\sum_{z_1\in\{0,1\}}(-1)^{x_1z_1}\left|z_1\right\rangle \frac{1}{\sqrt{2}}\sum_{z_1\in\{0,1\}}(-1)^{x_1z_1}\left|z_2\right\rangle ... \frac{1}{\sqrt{2}}\sum_{z_1\in\{0,1\}}(-1)^{x_1z_1}\left|z_n\right\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{z_1,z_2,...,z_n\in\{0,1\}}(-1)^{x_1z_1+x_2z_2+...+x_nz_n}\left|z_1\right\rangle\left|z_2\right\rangle ... \left|z_n\right\rangle
\end{aligned}$$