Lecture 4: Review of linear algebra and quantum mechanics
Reading: Kaye Chaps 2,3. Nielsen and Chuang Chap 2.

Since these basic topics are extensively covered in many other references included the quantum computing texts by Kaye et al and Nielsen and Chuang, this portion of the notes will just be a summary of the relevant material, details of which can be found in these references.

1 Linear algebra

Basic linear algebra knowledge is assumed. Key topics to be familiar with are tensor products, Schmidt decompositions, and a few others I list.

- Definition of Hilbert space, computational basis, dual vectors, inner and outer products
- Definition of orthonormal basis and associated properties (resolution of identity)
- Basis for linear operators, adjoint operators, unitary operators, projectors, spectral theorem
- Tensor products of vector spaces and linear operators on these spaces
- Schmidt decomposition

2 Quantum mechanics

- General properties of a two-level system, Bloch sphere representation (will be discussed in detail later)
- Unitary time-evolution of a closed system, Pauli operators and their properties, rotations about various axes
- Composition of systems using tensor product
- Two qubit gates that cannot be written as a tensor product, e.g. CNOT, CZ
- Measurement postulate, projective measurement, von Neumann measurement, expected values of observables
- Mixed states, density operators and their properties, measurements with density operators, representation on Bloch sphere
- Partial trace
- Superoperators as trace-preserving completely positive maps

3 A bit more on the Bloch sphere

3.1 Link between SO(3) and SU(2)

Many discussion of quantum states on the Bloch sphere work in the state vector picture in which a rotation matrix rotates a state vector about some axis, e.g. $R_z(\theta) = e^{-i\theta/2\sigma_z}$ rotations a ket $|\psi\rangle$ about the z axis. This perspective obscures a very important relation between rotations and the SU(2) and SO(3) groups. I will not go into great detail of the underlying group theory but just provide the bare minimum to get the sense of how rotations can be defined in each case and the relation to the Bloch sphere.

First, some preliminaries: SO(3) is the group of rotations acting about the origin on $\mathbb{R}^3$ with the product operation as composition. A representation of this group is 3×3 real orthogonal matrices Q.

SU(2) is the group of unit-determinant unitary matrices. This group possesses a Lie algebra su(2) that generates all elements of SU(2) by exponentiation. su(2) is a real vector space V_3 of traceless Hermitian matrices. The dimension of this vector space is 3 which is the same as $\mathbb{R}^3$, suggesting a link.

Let's define a scalar product for su(2) as

$$(X, Y) = \frac{1}{2} Tr(XY) \tag{3.1}$$

and consider the map $X \to X' = uXu^\dagger$. This map takes $V_3 \to V_3$ since (1) X' is Hermitian if X is; (2) $Tr(X') = 0$ if $Tr(X) = 0$; and (3) it preserves norms as rotations do, since

$$(X', X') = \frac{1}{2} Tr(uXu^\dagger uXu^\dagger) = (X, X) \tag{3.2}$$

Therefore, this map is actually a rotation on V_3 and should have a relation to rotations on $\mathbb{R}^3$! In V_3, the Pauli matrices are a basis and so any element is given by

$$X = \vec{x} \cdot \vec{\sigma} = \sum_i x_i \sigma_i \tag{3.3}$$

making clear the relation between V_3 and $\mathbb{R}^3$. We find that with this identification, operations on V_3 translate directly to those on $\mathbb{R}^3$, e.g.

$$(A, B) = \frac{1}{2} Tr[(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = \vec{a} \cdot \vec{b} \tag{3.4}$$

by the properties of Pauli matrices.

As remarked above, every element of SU(2) can be written as the exponential of a traceless anti-Hermitian operator as:

$$u(\theta) = e^{-i(\theta/2)\vec{e} \cdot \sigma} \tag{3.5}$$

With this expression, we can relate rotations in $\mathbb{R}^3$ to those in V_3. The steps are explicitly given in Frank Porter's notes, p12 and so I just quote the result here:

$$u_e(\theta) X u_e^\dagger(\theta) = X' = \vec{x}' \cdot \vec{\sigma} = R_e(\theta) \vec{x} \cdot \vec{\sigma} \tag{3.6}$$

where R_e is an SO(3) rotation matrix about axis $\vec{e}$.

Put another way,

$$u^\dagger \sigma_i u = R_e(\theta)_{ij} \sigma_j \tag{3.7}$$

It is a useful exercise to go through the steps of the above result on your own. Now let's move on to how this result is useful for interpreting the Bloch sphere representation of a quantum state. A general mixed quantum state can be expressed as a density operator:

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i| = \frac{1}{2}(I + \vec{x} \cdot \vec{\sigma}) \tag{3.8}$$

where the second equality is because any unit trace, Hermitian operator can be expressed with the expansion given. That result shows that we can associate a density matrix with a point in $\mathbb{R}^3$.

Let's consider a pure state for which the density operator is simply $\rho = |\psi\rangle \langle\psi|$. It has the property that $\rho^2 = \rho$. To make this requirement satisfied, we find:

$$\rho^2 = \frac{1}{4}(I + \vec{x} \cdot \vec{\sigma})(I + \vec{x} \cdot \vec{\sigma}) = \frac{1}{2}(I + \vec{x} \cdot \vec{\sigma}) \tag{3.9}$$

which you can figure out requires that $(\vec{x} \cdot \vec{\sigma})^2 = 1$, which in turn requires that $\|x\| = 1$ - or $\vec{x}$ describes a point on the unit sphere! This result is consistent with what is given in standard quantum computing books - a pure state is a point on the Bloch sphere.

So we now see that a unitary operation on ρ corresponds to a rotation of $\vec{x}$:

$$u\rho u^\dagger = \frac{1}{2}(I + \vec{x} \cdot (u\vec{\sigma}u^\dagger)) = \frac{1}{2}(I + (R_e(\theta)\vec{x}) \cdot \vec{\sigma}) \tag{3.10}$$

and $u = \exp(-i(\theta/2)\vec{e} \cdot \vec{\sigma})$ performs the rotation.

3.2 Relation to state vector picture

With all this work in mind, let's return to the standard description of the Bloch sphere using the state vector. A pure two-level state is typically described by enforcing that the norm of the state is unity and neglecting a global phase to get:

$$|\psi\rangle = \cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle \leftrightarrow \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} \tag{3.11}$$

Therefore, the density matrix for this pure state is:

$$\rho = \begin{pmatrix} c^2 & e^{-i\phi}cs \\ e^{i\phi}cs & s^2 \end{pmatrix} \tag{3.12}$$

where c and s are short for $\cos\theta/2$ and $\sin\theta/2$. We can express this state as an expansion in Pauli matrices using the inner product to get each coefficient. For example,

$$n_x = \frac{1}{2}Tr(\rho\sigma_x) = cs\cos\phi = \frac{1}{2}\sin\theta\cos\phi \tag{3.13}$$

A similar process for y and z yields

$$\rho = \frac{1}{2}(I + \sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y + \cos\theta\sigma_z) \tag{3.14}$$

which is a point on the unit sphere, just as we found before! Now, again from the last section we know that rotations of this state are performed as $\rho \to U\rho U^\dagger = U\left|\psi\right\rangle\left\langle\psi\right|U^\dagger$. So we see we could either form the density matrix ρ, then apply the SU(2) rotation, or apply U to our state vector $\left|\psi\right\rangle$ and form a new density matrix with the new state vector. If we do it the latter way, we don't have to get into the details of the SU(2) and SO(3) relationship, but at risk of obscuring some important details about the Bloch sphere. In my view, it is better to spend some time to really understand the meaning of the Bloch sphere in terms of the basis of Pauli matrices. Hopefully you now do!